Newtonian fluid flow in a slightly tapered tube

A fluid (of constant density ρ) is in incompressible, laminar flow through a tube of length L. The radius of the tube of circular cross section changes linearly from R_0 at the tube entrance ($z = 0$) to a slightly smaller value R_L at the tube exit ($z = L$).

Such a flow occurs when a lubricant flows in certain lubrication systems (It means high viscosity, and so low Reynolds number, Re~1).

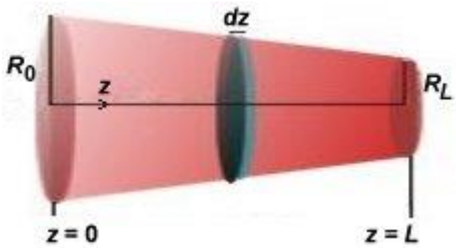

Figure. Fluid flow in a slightly tapered tube.

Using the lubrication approximation, determine the mass flow rate vs. pressure drop (w vs. ΔP) relationship for a Newtonian fluid (of constant viscosity μ).

Solution

Step. Hagen-Poiseuille equation and lubrication approximation:

The mass flow rate vs. pressure drop (w vs. ΔP) relationship for a Newtonian fluid in a circular tube of constant radius R is

$$w = \frac{\pi \, \Delta P \, R^4 \rho}{8 \mu L} \quad (1)$$

The above equation, which is the famous Hagen-Poiseuille equation, may be re-arranged as

$$\frac{\Delta P}{L} = \frac{8 \mu w}{\rho \pi} \frac{1}{R^4} \quad (2)$$

For the tapered tube, note that the mass flow rate w does not change with axial distance z. If the above equation is assumed to be approximately valid for a differential length dz of the tube whose radius R is slowly changing with axial distance z, then it may be re-written as

$$-\frac{dP}{dz} = \frac{8 \mu w}{\rho \pi} \frac{1}{[R(z)]^4} \quad (3)$$

The approximation used above where a flow between non-parallel surfaces is treated locally as a flow between parallel surfaces is commonly called the lubrication approximation because it is often employed in the theory of lubrication. The lubrication approximation, simply speaking, is a local application of a one-dimensional solution and therefore may be referred to as a quasi-one-dimensional approach.

Equation (3) may be integrated to obtain the pressure drop across the tube on substituting the taper function $R(z)$, which is determined next.

Step. Taper function

As the tube radius R varies linearly from R_0 at the tube entrance ($z = 0$) to R_L at the tube exit ($z = L$), the taper function may be expressed as $R(z) = R_0 + (R_L - R_0)\, z / L$. On differentiating with respect to z, we get

$$\frac{dR}{dz} = \frac{R_L - R_0}{L} \quad (4)$$

Equation (3) is readily integrated with respect to radius R rather than axial distance z. Using equation (4) to eliminate dz from equation (3) yields

$$(-\,dP) = \frac{8\,\mu\, w}{\rho\, \pi} \frac{L}{R_L - R_0} \frac{dR}{R^4} \quad (5)$$

Integrating the above equation between $z = 0$ and $z = L$, we get

$$\int_{P_0}^{P_L} (-dP) = \frac{8\,\mu\, w}{\rho\, \pi} \frac{L}{R_L - R_0} \int_{R_0}^{R_L} \frac{dR}{R^4} \quad (6)$$

$$\frac{P_0 - P_L}{L} = \frac{8\,\mu\, w}{\rho\, \pi} \frac{1}{3\,(R_L - R_0)} \left(\frac{1}{R_0^3} - \frac{1}{R_L^3} \right) \quad (7)$$

Equation (7) may be re-arranged into the following standard form in terms of mass flow rate:

$$w = \frac{\pi\, \Delta P\, R_0^4\, \rho}{8\,\mu\, L} \left[\frac{3\,(\lambda - 1)}{1 - \lambda^{-3}} \right] = \frac{\pi\, \Delta P\, R_0^4\, \rho}{8\,\mu\, L} \left[\frac{3\,\lambda^3}{1 + \lambda + \lambda^2} \right] \quad (8)$$

where the taper ratio $\lambda \equiv R_L / R_0$. The term in square brackets on the right-hand side of the above equation may be viewed as a taper correction to equation (1).